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Bunch Shape Evolution Near Transition, — an Intuitive Approach

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Abstract

The non-adiabatic time and the bunch shape at transition are derived from an intuitive approach. This derivation is designed for readers who wish to understand the physics but do not wish to go through the solution of a differential equation involving Bessel functions of fractional orders.

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I. INTRODUCTION

The evolution of a bunch near transition has been worked out in terms of Bessel functions of fractional orders by solving a differential equation. [1] However, all these can be estimated easily without going into differential equations and Bessel functions. Best of all, through the estimation, one can have a clear picture of what is going on during transition.

Here, we list the equations of motion of a bunch particle in the longitudinal phase space. A detailed derivation is given in Appendix. The slippage factor, defined as

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \quad (1.1)$$

where $E = \gamma E_0$ is the total energy of the synchronous particle having rest energy E_0 , and $\gamma_t E_0$ is the transition energy of the lattice. Physically, η measures the amount of time or phase slippage of a bunch particle with respect to the synchronous particle. Thus, for a particle with energy deviation ΔE , its rf phase slips at a rate of

$$\frac{d\Delta\phi}{dt} = \frac{h\eta\omega_0}{\beta^2 E} \Delta E, \quad (1.2)$$

where h is the rf harmonic, $\omega_0/2\pi$ the revolution frequency of the synchronous particle and β its velocity relative to c , the velocity of light. At the same time, this off-energy particle receives additional energy from the rf cavities at the rate of

$$\frac{d\Delta E}{dt} = \frac{eV\omega_0}{2\pi} [\sin(\phi_s + \Delta\phi) - \sin\phi_s], \quad (1.3)$$

where V is the rf voltage, ϕ_s and e the synchronous rf phase and charge of the synchronous particle. Equations (1.2) and (1.3) form the equations of motion of a bunch particle in the longitudinal phase space. Eliminating ΔE , we obtain for small $\Delta\phi$ the equation governing the motion of the phase of the particle:

$$\frac{d}{dt} \left(\frac{1}{\omega_s^2} \frac{d\Delta\phi}{dt} \right) + \Delta\phi = 0, \quad (1.4)$$

where

$$\omega_s = \sqrt{-\frac{eh\eta V \cos\phi_s}{2\pi\beta^2 E}} \omega_0. \quad (1.5)$$

When the inverse of the slippage factor, η^{-1} , does not change rapidly, we see from Eq. (1.4) that the particle performs synchrotron oscillations with a synchrotron tune $\nu_s = \omega_s/\omega_0$.

II. NON-ADIABATIC TIME

When η^{-1} is not rapidly changing, a bucket can be defined. The bucket height has the property (see Appendix),

$$(\Delta E)_{\text{bucket}} \propto \left(\frac{E}{|\eta|} \right)^{1/2}. \quad (2.1)$$

However, as the bunch particle passes through transition, η^{-1} changes rapidly. Let time t be measured from the moment transition is crossed or when $\gamma = \gamma_t$, and assume that the rf voltage and the synchronous phase, aside from flipping from ϕ_s to $\pi - \phi_s$, are held fixed near transition. Then, from Eq. (1.1),

$$\eta = \frac{2\dot{\gamma}t}{\gamma^3}. \quad (2.2)$$

This means that when transition is approached, synchrotron frequency is slowed down to zero and the bucket height is increased to infinity. In other words, when it is close enough to transition, the particle will not be able to catch up with the rapid changing of the bucket shape. This time period, from $t = -T_c$ to $t = T_c$ is called the non-adiabatic region, and T_c the non-adiabatic time. Here, we define this region by

$$\omega_s \leq \frac{2}{(\Delta E)_{\text{bucket}}} \frac{d(\Delta E)_{\text{bucket}}}{dt}. \quad (2.3)$$

The right side is

$$\frac{1}{(\Delta E)_{\text{bucket}}} \frac{d(\Delta E)_{\text{bucket}}}{dt} \Big|_{t=-T_c} = \frac{\dot{\gamma}}{2\gamma} - \frac{\dot{\eta}}{2\eta} = \frac{\dot{\gamma}}{2\gamma} + \frac{1}{2T_c}. \quad (2.4)$$

At $t = -T_c$, $|\Delta\gamma| = \dot{\gamma}T_c$, the deviation of γ from γ_t , is usually less than one unit and is therefore much less than γ_t . Therefore, the term $\dot{\gamma}/\gamma$ in Eq. (2.4) can be neglected, and inside the non-adiabatic region it is a good approximation to replace the relativistic factors β and γ by their values right at transition, denoted by the subscript t . Evaluating at $t = -T_c$, the left side of Eq. (2.3) is

$$\omega_s|_{t=-T_c} = \sqrt{\frac{h\dot{\gamma}_t T_c e V \cos \phi_s}{\pi \beta_t^2 \gamma_t^4 E_0}} \omega_0. \quad (2.5)$$

We then obtain the non-adiabatic time from Eq. (2.3):

$$T_c = \left[\left(\frac{\beta_t^2 \gamma_t^4}{2\omega_0 h} \right) \left(\frac{|\tan \phi_s|}{\dot{\gamma}_t^2} \right) \right]^{1/3}, \quad (2.6)$$

where

$$\dot{\gamma}_t = \frac{eV\omega_0}{2\pi E_0} \sin \phi_s \quad (2.7)$$

derived from Eq. (1.3) has been used. Note that the non-adiabatic time is just an approximate time. The factor 2 on the right side of Eq. (2.3) was inserted for the purpose that T_c given by Eq. (2.6) is exactly the same as the adiabatic time quoted in the literature. We have written Eq. (2.6) in such a way that the factor in the first brackets contains parameters of the lattice, while $\dot{\gamma}_t$ in the second brackets is determined by the ramp curve and ϕ_s , the synchronous phase at transition, is determined by the rf-voltage table.

III. BUNCH SHAPE AT TRANSITION

For the sake of simplicity, we adopt a model which states that,

- (1) when $|t| > T_c$, the beam particles follow the bucket with synchrotron oscillations, and
- (2) when $|t| < T_c$, the beam particles make no synchrotron oscillations at all.

From Eq. (1.2), the bunch length σ_ϕ at $t = -T_c$ is related to the rms energy spread σ_E by

$$\nu_s \sigma_\phi = \frac{h\eta}{\beta_t^2 \gamma_t E_0} \sigma_E, \quad (3.1)$$

where η is to be evaluated at $t = -T_c$, and the energy E is evaluated approximately right at transition since the change is slow. The 95% bunch area is defined as

$$S = 6\pi \sigma_\tau \sigma_E. \quad (3.2)$$

From Eqs. (3.1) and (3.2), we obtain the rms bunch length in time $\sigma_\tau = h\omega_0 \sigma_\phi$ as

$$\sigma_\tau = \left(\frac{S\eta}{6\pi\nu_s\omega_0\beta_t^2\gamma_tE_0} \right)^{1/2}. \quad (3.3)$$

Substituting $\eta(-T_c)$ from Eq. (2.2) and $\dot{\gamma}_t$ from Eq. (2.7), we arrive at

$$\sigma_\tau = \frac{1}{\sqrt{\pi}} \left(\frac{ST_c^2 \dot{\gamma}_t}{\beta_t^2 \gamma_t^4 E_0} \right)^{1/2}. \quad (3.4)$$

Our simple model requires no synchrotron oscillation inside the non-adiabatic region. This is equivalent to having $\eta = 0$ in Eq. (1.3); or the phase of each particle will not change at all. Therefore, Eq. (3.4) is also the bunch length right

at transition, which agrees with the result from solving a differential equation if we make the replacement

$$\frac{1}{\sqrt{3\pi}} = 0.326 \quad \Rightarrow \quad \frac{2}{3^{5/6}, (\frac{1}{3})} = 0.300 , \quad (3.5)$$

where , $(\frac{1}{3})$ is the Gamma function. Our estimate is about 8.8% too large because our simple model does not allow the bunch to continue to shrink in the non-adiabatic region.

On the other hand, without synchrotron oscillations, the energy of each beam particle is accelerated by the focusing rf force according to Eq. (1.3). From $t = -T_c$ to $t = 0$, a particle at a phase offset $\Delta\phi$ from the synchronous particle will acquire an energy

$$\Delta E = T_c E_0 \frac{d\dot{\gamma}}{d\Delta\phi} \Delta\phi , \quad (3.6)$$

where

$$\frac{d\dot{\gamma}}{d\Delta\phi} \approx \frac{\dot{\gamma}_t}{\tan\phi_s} , \quad (3.7)$$

and the small phase-offset approximation has been made. But at $t = -T_c$ the energy spread at $\Delta\phi$ of the bunch is given by Eq. (3.1):

$$\Delta E = \frac{\nu_s \beta_t^2 \gamma_t E_0}{h\eta} \sqrt{(\Delta\phi)_c^2 - (\Delta\phi)^2} , \quad (3.8)$$

where $(\Delta\phi)_c = \sqrt{6}\sigma_\tau h\omega_0$ is the half width of the bunch at $t = -T_c$ as given by Eq. (3.4). When evaluated at $t = -T_c$, it is found that the coefficient of Eq. (3.8) is equal to that of Eq. (3.6), and we denote it by

$$a = \frac{\nu_s \beta_t^2 \gamma_t E_0}{h\eta} = T_c E_0 \frac{d\dot{\gamma}}{d\Delta\phi} . \quad (3.9)$$

Therefore, the total energy spread at transition is given by

$$(\Delta E)_{\text{total}} = a \left[\sqrt{(\Delta\phi)_c^2 - (\Delta\phi)^2} + \Delta\phi \right] . \quad (3.10)$$

The maximum total energy spread comes out to be

$$(\Delta E)_{\text{total,max}} = \frac{1}{\sqrt{\pi}} \left(\frac{S \beta_t^2 \gamma_t^4 E_0}{T_c^2 \dot{\gamma}_t} \right)^{1/2} . \quad (3.11)$$

at $\Delta\phi = 2^{-1/2}(\Delta\phi)_c$. The exact value from the solution of a differential equation can be obtained from the replacement

$$\frac{1}{\sqrt{\pi}} = 0.564 \quad \Rightarrow \quad \frac{, (\frac{1}{3})}{3^{1/6} 2^{1/2} \pi} = 0.502 . \quad (3.12)$$

By the same token, the particle at the tail of the bunch will be decelerated by the same energy. Particles in between will be accelerated accordingly. The bunch shape at transition is therefore given by Fig. 1, which is slanted at an angle from the ΔE -axis.

As shown in Eq. (3.12), our estimate of $(\Delta E)_{\text{total}}$ is about 11% too large. This is to be expected because we allow pure increment in energy by the focusing rf potential in the non-adiabatic region without any motion in the phase direction.

To conclude this section, let us write the rms time spread and rms energy spread at transition as well as the non-adiabatic time in terms of the quantities that we can control, namely, the synchronous phase ϕ_s and ramping rate $\dot{\gamma}_t$:

$$\sigma_\tau \propto \frac{\tan^{\frac{1}{3}} \phi_s}{\dot{\gamma}_t^{\frac{1}{6}}}, \quad \sigma_E \propto \frac{\dot{\gamma}_t^{\frac{1}{6}}}{\tan^{\frac{1}{3}} \phi_s}, \quad T_c \propto \frac{\tan^{\frac{1}{3}} \phi_s}{\dot{\gamma}_t^{\frac{2}{3}}}. \quad (3.13)$$

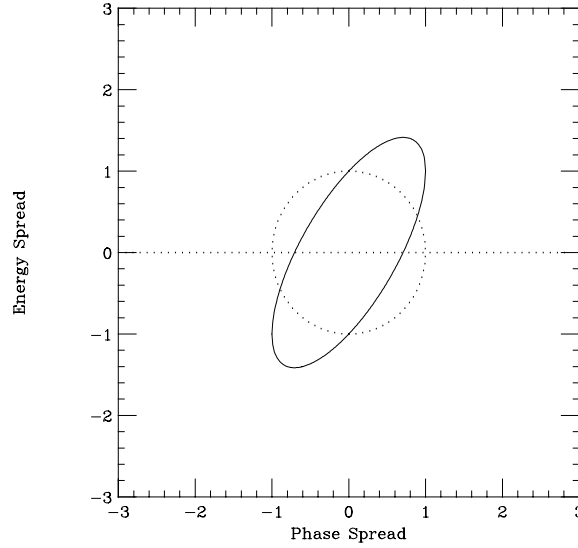


Figure 1: The evolution of the bunch, according to the simple model, from $t = -T_c$ (dots) to the time when transition is crossed (solid).

IV. MORE SOPHISTICATED APPROXIMATION

We now discard the simple model in the previous section and come back to Eq. (1.4), the equation governing motion of small phase offset. In the adiabatic region that is not too far away from transition, the particle is performing synchrotron oscillations with a slowly changing frequency $\omega_s/2\pi$ given by Eq. (1.5). The solution of Eq. (1.4) is therefore of the form

$$\Delta\phi = Ae^{i\int\omega_s dt}, \quad (4.1)$$

where the amplitude A is also slowly changing. We then have

$$\frac{d}{dt}\left(\frac{1}{\omega_s^2}\frac{d\Delta\phi}{dt}\right) = -\Delta\phi + \left[\left(\frac{2i\dot{A}}{\omega_s} - \frac{iA\dot{\omega}_s}{\omega_s^2}\right) + \left(\frac{\ddot{A}}{\omega_s^2} - \frac{2\dot{A}\dot{\omega}_s}{\omega_s^3}\right)\right]e^{i\int\omega_s dt}. \quad (4.2)$$

Since $\Delta\phi$ varies much faster than A and ω_s , we can neglect \ddot{A} , $\ddot{\omega}_s$, and $\dot{A}\dot{\omega}_s$, and set

$$\frac{2\dot{A}}{\omega_s} = \frac{A\dot{\omega}_s}{\omega_s^2}, \quad (4.3)$$

so that Eq. (1.4) is satisfied. The relation in Eq. (4.3) leads to

$$\frac{A^2}{\omega_s} = \text{constant}, \quad (4.4)$$

implying that the solution of Eq. (1.4) or the rf phase of a beam particle in the adiabatic region can be written as

$$\Delta\phi = B\sqrt{\omega_s}e^{i\int\omega_s dt}, \quad (4.5)$$

with B being constant.

The dropping of the slowly varying terms from Eq. (4.2) is equivalent to assuming

$$\frac{\ddot{A}}{\omega_s^2} \ll \frac{A\dot{\omega}_s}{\omega_s^2}, \quad (4.6)$$

$$\frac{2\dot{A}\dot{\omega}_s}{\omega_s^3} \ll \frac{A\dot{\omega}_s}{\omega_s^2}. \quad (4.7)$$

Again we assume constant rf voltage V and constant synchronous phase ϕ_s , and obtain, similar to Eq. (2.2),

$$\omega_s^2(t) = b|t| \quad \text{with} \quad b = \frac{\dot{\gamma}_t h e V |\cos \phi_s| \omega_0^2}{\pi \beta_t^2 \gamma_t^4 E_0}. \quad (4.8)$$

Then, together with Eq. (4.4),

$$\text{Eq. (4.6)} \quad \Rightarrow \quad |t| \gg \left(\frac{1}{2}\right)^{2/3} \left(\frac{1}{b}\right)^{1/3}, \quad (4.9)$$

$$\text{Eq. (4.7)} \quad \Rightarrow \quad |t| \gg \left(\frac{3}{8}\right)^{2/3} \left(\frac{1}{b}\right)^{1/3}, \quad (4.10)$$

A non-adiabatic time T_c can therefore be defined. If we let

$$T_c = \left(\frac{1}{b}\right)^{1/3}, \quad (4.11)$$

it just turns out to be exactly the same expression in Eq. (2.4). Here, we arrive at a neat way to remember the non-adiabatic time:

$$\omega_s^2 = \frac{|t|}{T_c^3} \quad \text{or} \quad \omega_s|_{t=-T_c} = \frac{1}{T_c}. \quad (4.12)$$

Now, let us continue the study of the bunch shape in the adiabatic region. Differentiating Eq. (4.5) and using Eq. (4.4), we get

$$\frac{d\Delta\phi}{dt} = iB\omega_s^{3/2} \left[1 - \frac{i}{4} \left(\frac{T_c}{|t|}\right)^{3/2} \right] e^{\int \omega_s dt}, \quad (4.13)$$

or

$$\frac{d\Delta\phi}{dt} = i\omega_s\Delta\phi \left[1 + \frac{1}{16} \left(\frac{T_c}{|t|}\right)^3 \right]^{1/2} e^{i\varphi}, \quad (4.14)$$

with

$$\varphi = \tan^{-1} \frac{1}{4} \left(\frac{T_c}{|t|}\right)^{3/2}. \quad (4.15)$$

Then, using Eq. (1.2), we arrive at the energy offset of the particle

$$\Delta E = -i\omega_s\Delta\phi \frac{\beta^2\gamma E_0}{|\eta|h\omega_0} \left[1 + \frac{1}{16} \left(\frac{T_c}{|t|}\right)^3 \right]^{1/2} e^{i\varphi}, \quad (4.16)$$

We see from Eq. (4.5) that, as the bunch is approaching the non-adiabatic region, its width shrinks in the same way as the decrease of $\sqrt{\omega_s}$. On the other hand, from Eq. (4.16), the height of the bunch increases because of the square root term and the $t^{-1/4}$ dependency in the front factor. We also see that there is a phase advance φ of the energy offset, or a tilt in the bunch shape in the longitudinal phase space. This tells us that there is already slowing down in the phase motion in the adiabatic region when transition is approached. In other

words, there is no clear cut boundary between the adiabatic and non-adiabatic regions.

The next task is to relate the constant B to the bunch area. The motion of the particle described by Eqs. (4.5) and (4.6) is of the form

$$\Delta\phi = \mathcal{A} \cos \theta , \quad \Delta E = \mathcal{B} \sin(\theta + \varphi) , \quad (4.17)$$

which map out a tilted ellipse of area $\pi \mathcal{A} \mathcal{B} \cos \varphi$ inscribed inside the rectangle of width $2\mathcal{B}$ and height $2\mathcal{A}$. Therefore the bunch area in eV-s is

$$S = \frac{B^2 e V \cos \phi_s}{2h} \quad (4.18)$$

which is time independent as anticipated. In above, the expression of ω_s as given by Eq. (1.5) has been used.

The motion of a particle in the non-adiabatic region can also be studied using Eq. (1.4), which indicates an expansion into a double series:

$$\Delta\phi = \sum_{n=0}^{\infty} a_n \left(\frac{|t|}{T_c} \right)^{3n} + \frac{|t|^2}{T_c^2} \sum_{n=0}^{\infty} b_n \left(\frac{|t|}{T_c} \right)^{3n} , \quad (4.19)$$

where a_0 and b_0 are determined, respectively, by the bunch length and bunch height at transition. The analysis is rather involved and we are not going to pursue it further in this paper.

APPENDIX

A. Slippage Factor η and Transition Crossing

Let us first review some basic stuff. The most important variable here is the slippage factor η , which is defined in terms of the time delay ΔT of a particle which has an off-momentum δ with respect to the synchronous particle: [2]

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C_0} - \frac{\Delta v}{v_0} = \eta \delta , \quad (A.1)$$

where T_0 is the period of the synchronous particle, C_0 its orbit length, and v_0 its velocity. This just states that, being off-momentum, the particle may be traveling on a longer orbit giving rise to the time delay. However, the larger velocity can give rise to a time advance instead. We have

$$\frac{\Delta C}{C_0} = \alpha_0 \delta \quad (A.2)$$

where α_0 is a property of the lattice and is called the *momentum-compaction* factor, while

$$\frac{\Delta v}{v_0} = \frac{\delta}{\gamma^2} \quad (\text{A.3})$$

where γE_0 is the energy of the particle with E_0 being its rest energy. The slippage factor can therefore be written as

$$\eta = \left(\alpha_0 - \frac{1}{\gamma^2} \right) \delta = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \right) \delta \quad (\text{A.4})$$

where $\gamma_t = 1/\sqrt{\alpha_0}$ is called the transition gamma and $E_t = \gamma_t E_0$ is the transition energy. We see that when the particle energy E is below E_t , or the relativistic factor $\gamma < \gamma_t$, the kinematic term in Eq. (A.1) or (A.4) dominates over the effect of orbit-length increase. Therefore, η is negative implying that the particle with an off momentum $\delta > 0$ will revolve faster than the synchronous particle and arrive ahead after a turn. In order that the synchrotron oscillation is stable, it is required to restrict the synchronous phase to $-\frac{1}{2}\pi < \phi_s < \frac{1}{2}\pi$, so that ω_s in Eq. (1.5) is real. As the bunch particles are ramped, the kinematic term decreases because of the finite limitation of the velocity of light and at some time is canceled by the term for orbit-length increase, resulting in $\eta \rightarrow 0$. This is the moment we call transition crossing, and all off-momentum particles have exactly the same revolution period as the synchrotron particle. As the energy is further increased, the effect of going on a longer orbit dominates and η becomes positive. For stable synchrotron motion, we must have now $\frac{1}{2}\pi < \phi_s < \frac{3}{2}\pi$. This explains why there must be a jump of the synchronous phase at transition.

It is important to point out that, even if the momentum-compaction factor is a constant independent of momentum offset, particles inside a bunch will actually cross transition at different times. For example, particles with positive momentum offsets have higher energy than the synchronous particle, and will therefore reach transition earlier. Because the synchronous phase can only be jumped at one time, many particles in the bunch will experience unstable synchrotron motion for a short period, resulting in emittance growth, beam loss, and other nasty phenomena. In an actual machine, the momentum-compaction factor is usually not independent of momentum offset. This may reduce or enhance the difficulties of transition crossing.

B. Equations of Motion

From the n -th turn to the $(n+1)$ -th turn, the change in rf phase offset with respect to the synchronous particle is given by

$$\Delta\phi_{n+1} = \Delta\phi_n + \frac{2\pi h\eta}{\beta^2\gamma E_0} \Delta E_n , \quad (\text{B.1})$$

where η , β , and γ are of the synchronous particle at the n -th turn. The energy offset is updated by the rf voltage:

$$\Delta E_{n+1} = \Delta E_n + eV [\sin(\phi_s + \Delta\phi_{n+1}) - \sin\phi_s] . \quad (\text{B.2})$$

Since the changes per turn are small, Eqs. (B.1) and (B.2) can be smoothed out to form a set of differential equations:

$$\frac{d\Delta\phi}{dt} = \frac{h\eta\omega_0}{\beta^2 E} \Delta E , \quad (\text{B.3})$$

$$\frac{d\Delta E}{dt} = \frac{eV\omega_0}{2\pi} [\sin(\phi_s + \Delta\phi) - \sin\phi_s] , \quad (\text{B.4})$$

which forms the equations of motion of a particle in the longitudinal phase space. There is a constant of motion,

$$H = \frac{h\eta\omega_0}{2\beta^2 E} (\Delta E)^2 + \frac{eV\omega_0}{2\pi} [\cos(\phi_s + \Delta\phi) - (\phi_s - \Delta\phi) \sin\phi_s] , \quad (\text{B.5})$$

which is the Hamiltonian. The fixed points are obtained by solving $d\Delta\phi/dt = d\Delta E/dt = 0$. One solution, $\Delta E = 0$ and $\Delta\phi = \phi_s$, is the stable fixed point, around which the particle performs synchrotron oscillations. The other solution,

$$\begin{aligned} \Delta E &= 0 \\ \Delta\phi &= \pi - 2\phi_s , \end{aligned} \quad (\text{B.6})$$

is the unstable fixed point, through which the bucket boundary or separatrix passes. Substituting the unstable fixed point into Eq. (B.5), the value of the Hamiltonian at the separatrix can be evaluated. We can then solve from the Hamiltonian the bucket height

$$(\Delta E)_{\text{bucket}} = \left\{ \frac{eV\beta^2 E}{\pi h\eta} [-2\cos\phi_s + (\pi - 2\phi_s) \sin\phi_s] \right\}^{1/2} . \quad (\text{B.7})$$

References

- [1] See, for example, S.Y. Lee and J.M. Wang, *Microwave Instability Across the Transition Energy*, IEEE Trans. Nucl. Sc. **NS32**, 2323 (1985).
- [2] This definition of the slippage factor is equivalent to that in Eq. (1.2) when η does not depend on momentum offset. If higher orders are included, Eq. (1.2) should be used instead. See, for example, K.Y. Ng, *Muon Bunch inside a Quasi-Isochronous Bucket*, Fermilab Report FN-645, (1996).